AN INTEGRATED KIRCHHOFF ELEMENT FOR THE ANALYSIS OF PLATES ON ELASTIC FOUNDATION

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Abstract— Plates supported on elastic foundations are encountered in many Engineering applications. Conventionally such systems can be analysed using regular plate bending element plus discrete soil springs. The present work aims at an element formulation suitable for analysis of such systems without the use of explicit discrete soil springs. The scope of the work includes static analysis of an isotropic rectangular plate resting on elastic foundation with various boundary conditions, various types of load applications for varying properties of foundation. In this paper, finite element analysis has been carried out for an isotropic rectangular plate by using a four noded Kirchhoff rectangular element with four degrees of freedom per node, with Winkler model for Elastic foundation. The finite element formulation has been carried out by integrating the properties of the plate with those of elastic foundation using Galerkin's approach instead of the commonly used potential energy approach. Numerical analysis has been carried out by suitable MATLAB code and the results obtained are in good agreement with those reported in earlier studies.

Keywords— Elastic foundation, Galerkin's method, Kirchhoff theory, Rectangular plate, Winkler model

I. INTRODUCTION

Plates on elastic foundation have wide application in engineering such as foundations, storage tanks, swimming pools, floor systems of buildings and highways and airfield pavements etc. The complexity of the analytical formulations of plates on elastic foundation limited the number of available analytical solutions. So the need for robust and versatile numerical solutions has become critical. Several numerical methods have been used by researchers to solve the plate-bending problem. These numerical methods include finite difference method, Ritz method, finite strip method, boundary element method and the finite element method. Among the numerical methods mentioned above the finite element method is the most versatile one. The field of plate bending has been an area of intensive research since the introduction of the finite element method in the early 1960s and still remains to be one of the active research fields. This is, mainly, due to the wide application of plate elements in engineering as indicated above and also due to the complexity of modelling plate elements. The complexity of modelling plate elements generally stem from the difficulties of obtaining suitable shape functions that preserve strain or slope continuity and satisfying the compatibility conditions in the case of thin plates. Also, failure of formulations based on thick plate theory to give good results when plate thickness becomes small is another daunting problem that haunted the development of successful thick plate bending elements. In spite of that, there is a large number of plate elements developed which fall into the various categories of non-conforming (non compatible), conforming (compatible), etc. Many of these elements were successfully used in practice.

The mechanical modelling of plate-subsoil interaction problem is mathematically quite complex phenomenon and the response of subgrade is governed by many factors. A simple and widely used one is Winkler model where it is assumed that the foundation soil consists of linear elastic springs and each spring is independent of the others. Generally, analysis of the bending of plates on an elastic foundation is developed on the assumption that the reaction forces of the foundation are proportional at every point to the deflection of the plane at that point. This assumption was first introduced by Winkler for the analysis of railroad tracks. The difficulty with the Winkler model applied for analysis of plates on elastic foundations is the necessity of the evaluation of the modulus of the subgrade reaction k_s, which does not have a unique value for a particular soil or a particular loading on the plate. However, the Winkler model has been used for everyday design by practicing engineers because of its simplicity.

II. MODELLING THE BEHAVIOR OF PLATES

A. General

Plates are structures with very small thickness compared to its planar dimensions. Slabs in civil engineering structures, bearing plates under columns, parts of mechanical components, etc. are common examples of plates. The bending properties of a plate depend greatly on its thickness. Hence, in classical theory we have the following groups, viz: (i) thin plates with small deflections, (ii) thin plates with large deflections and (iii) thick plates [1]-[3]. There are mainly three theories of plate analysis. Namely: Kirchhoff or Classical Plate Theory (thin plates), Mindlin or thick Plate theory also known as First Order Shear deformation Theory (thick plates) and Third Order Shear Deformation Theory (laminates).The most widely used plate theory is classical Kirchhoff thin plate theory which ignores the effect of the shear deformation through plate thickness. The basic assumptions being considered under classical Kirchhoff's plate bending theory are identical to the Euler-Bernoulli beam theory assumptions. The following assumptions are considered with the Kirchhoff theory:

a) There is no deformation in the middle plane of the plate. This plane remains neutral during bending.

b) Points of the plate lying initially on a normal to the middle surface of the plate remain on the normal to the same surface even after bending.

c) The normal stresses in the direction transverse to the plate are negligible. However, the effect of the shear deformation becomes important as the thickness of plate increases. For this reason, it is obvious that shear deformations have to be taken into account especially for thick plates [4]. Mindlin plate element that includes the effect of shear deformation is fundamentally simple to adopt for analysis of plates on elastic foundation [5]- [8]. However, Mindlin plate elements cause shear locking when the plate becomes thin.

B. Kirchhoff plate element with sixteen degrees of freedom Kirchhoff or Classical Plate Theory is used to model the plate. In these elements C^2-continuity is considered, i.e. at each of the four nodes, four degrees of freedom, namely w, $\partial w/\partial x$,$\partial w/\partial y$ and $\partial^2 w/\partial x\partial y$ are treated as basic unknowns. Hence it leads to 16 degrees of freedom per element. This type of element is shown in Fig. 1. The typical element has size 2a × 2b. If N_i is the shape function at node i and i= 1 to 4. w_i and θ_i are the displacement and rotations The displacement field for any point can be expressed as

$$w = \Sigma N_{i1}w_i + \Sigma N_{i2}\theta_{xi} + \Sigma N_{i3}\theta_{yi} + \Sigma N_{i4}\theta_{xyi} \quad (1)$$

The shape functions in terms of natural coordinates ξ andη are given by [9]

$$N_{11}(\xi,\eta) = \psi_1^0(\xi)\psi_1^0(\eta)$$
$$N_{12}(\xi,\eta) = \psi_1^1(\xi)\psi_1^0(\eta)$$
$$N_{13}(\xi,\eta) = \psi_1^0(\xi)\psi_1^1(\eta)$$
$$N_{14}(\xi,\eta) = \psi_1^1(\xi)\psi_1^0(\eta)$$
$$N_{21}(\xi,\eta) = \psi_2^0(\xi)\psi_1^0(\eta)$$
$$N_{22}(\xi,\eta) = \psi_2^1(\xi)\psi_1^0(\eta)$$
$$N_{23}(\xi,\eta) = \psi_2^0(\xi)\psi_1^1(\eta)$$

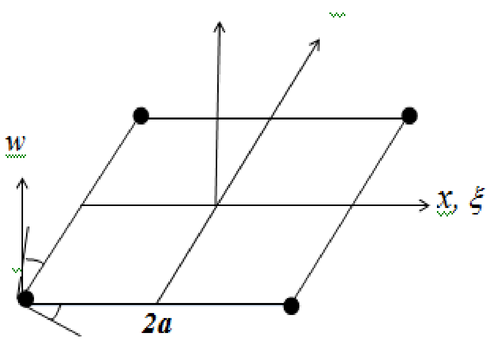

Figure1. Four –noded rectangular finite element used in this study

$$N_{24}(\xi,\eta) = \psi_2^1(\xi)\psi_1^1(\eta)$$
$$N_{31}(\xi,\eta) = \psi_2^0(\xi)\psi_2^0(\eta)$$
$$N_{32}(\xi,\eta) = \psi_2^1(\xi)\psi_2^0(\eta)$$
$$N_{33}(\xi,\eta) = \psi_2^0(\xi)\psi_2^1(\eta)$$
$$N_{34}(\xi,\eta) = \psi_2^1(\xi)\psi_2^1(\eta)$$
$$N_{41}(\xi,\eta) = \psi_1^0(\xi)\psi_2^0(\eta)$$
$$N_{42}(\xi,\eta) = \psi_1^1(\xi)\psi_2^0(\eta)$$
$$N_{43}(\xi,\eta) = \psi_1^0(\xi)\psi_2^1(\eta)$$
$$N_{44}(\xi,\eta) = \psi_1^1(\xi)\psi_2^1(\eta)$$

For any $(\xi,\eta)\in[-1,+1]$, where $\psi_1^0,\psi_1^1,\psi_2^0,\psi_2^1$ are the cubic Hermite polynomials defined on [-1,+1]

$$\psi_1^0(\xi) = ¼(\xi-1)^2(\xi+2), \qquad \psi_2^0(\xi) = ¼(\xi+1)^2(2-\xi)$$
$$\psi_1^1(\xi) = ¼(\xi-1)^2(\xi+1), \qquad \psi_2^1(\xi) = ¼(\xi+1)^2(\xi-1)$$

III. MODELLING THE BEHAVIOR OF WINKLER FOUNDATION

The effect of a foundation can be modeled by various approaches on the plate. The best realistic model is to represent the foundation as a continuum model where the elasticity solution represents the behavior of the foundation. On the other hand, the elastic foundation can be modeled as a set of springs. The simplest model presented for the elastic foundation is the Winkler model. Winkler model assumes that shear resistance of the foundation is ignorable compared to the shear capacity of foundation and models the foundation as a set of independent springs. Therefore, there is no lateral interaction between the springs. The hurdle with the Winkler model applied for analysis of plates on elastic foundations is the necessity of the evaluation of the modulus of the subgrade reaction, k_s, which does not have a unique value for a particular soil or a particular loading on the plate. The main disadvantages of this model are the discontinuity in the soil displacement between the soil under the structure and that outside the structure. Winkler model gives a constant displacement of the plate for a uniformly distributed load which results in a zero bending moment and shear force in the plate, thus creating non-conservative design criteria. However, the Winkler model has been used for everyday design by practicing engineers because of its simplicity.

IV. FORMULATION OF THE INTEGRATED FINITE ELEMENT BY GALERKIN'S METHOD

The finite element formulation is done by integrating the properties of the plate with those of elastic foundation using Galerkin's approach instead of the commonly used potential energy approach. The Galerkin method is extended to solving the plate equation of plate on elastic foundation. From Plate theory, if w is the displacement and k_s is the modulus of subgrade reaction of soil, the equilibrium under an applied vertical loading of intensity q demands:,

$$D\nabla^4 w + k_s w = q \quad (2)$$

$$D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) + k_s w = q \quad (3)$$

The approximate solution $\tilde{w}$ is of the form

$$\tilde{w} = [N]\{\Delta\} \quad (4)$$

Substituting this in Galerkin Criterion on weighted residual
yields:

$$\int_A \{N\}^T\{D(\nabla^4\tilde{w}) - k_s\tilde{w}\}dxdy = 0$$

$$D\int_A \{N\}^T\{D(\nabla^4\tilde{w})dxdy = k_s\int_A \{N\}^T\tilde{w}dxdy \quad (5)$$

V. NUMERICAL ANALYSIS AND RESULTS

Numerical modelling of the plates on Winkler foundation has been carried out with the above integrated finite element in a MATLAB environment, and results obtained are compared with those available in literature for verification. When the value of non-dimensional soil stiffness K is zero then the structure is equivalent to an ordinary plate, for which the exact solution is available. Comparing, the model is found to be in good agreement with these exact values, which validates the numerical MATLAB coding. Plates on Winkler foundation for different support conditions, different loading conditions are modeled for different values of non-dimensional soil stiffness K. The results obtained for these are comparable with previous studies [10]-[13]. and a comparison is made here with Mishra and Chakrabarty [14], O¨zgan and Dalo˘glu [15], Y.I. O¨zdemir[16]-[17]. as given in Figs. 2 to 5 and Tables 1 to 4. The central deflection of the structure is used for comparison in all cases. The element used in present study compares well with PBQ8, MT8 andnMT17 which are higher order elements.

Table1.Non-dimensional central displacements for the clamped plates with uniformly distributed load

K	Mishra and Chakrabarti	O¨zgan and Dalo˘glu PBQ4	O¨zgan and Dalo˘glu PBQ8	Y.I. O¨zdemir MT8	Y.I. O¨zdemir MT17	Present study
0	0.136	0.1228	0.1369	0.1369	0.1372	0.1334
3	0.127	0.1154	0.1277	0.1277	0.128	0.1247
6	0.062	0.0596	0.0622	0.0622	0.0622	0.0621
9	0.017	0.0173	0.0172	0.0172	0.0172	0.0175

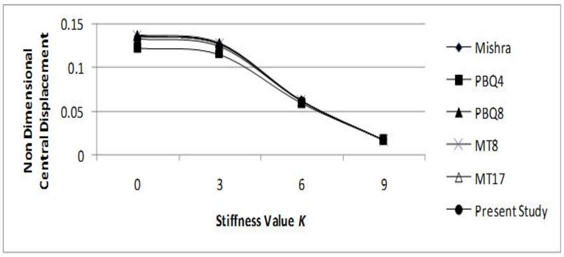

Figure 2. Non-dimensional central displacement of clamped plates with different K , subjected to uniformly distributed load

Table2.Non-dimensional central displacements for the clamped plates with concentrated load

K	Mishra and Chakrabarti	O¨zgan and Dalo˘glu PBQ4	O¨zgan and Dalo˘glu PBQ8	Y.I. O¨zdemir MT8	Y.I. O¨zdemir MT17	Present study
0	0.654	0.5914	0.6507	0.6509	0.6419	0.62
3	0.623	0.5656	0.6186	0.6188	0.6097	0.5876
6	0.392	0.3667	0.3854	0.3855	0.3761	0.3489
9	0.214	0.2006	0.2062	0.2063	0.197	0.1631

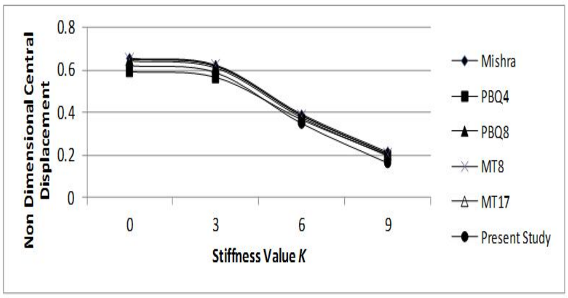

Figure 3. Non-dimensional central displacement of clamped plates with different K values subjected to concentrated load

Table 3. Non-dimensional central displacements for the simply supported plates with uniformly distributed load

K	Mishra and Chakrabarti	O¨zgan and Dalo˘glu PBQ4	O¨zgan and Dalo˘glu PBQ8	Y.I. O¨zdemir MT8	Y.I. O¨zdemir MT17	Present study
0	4.13	3.8487	4.1539	4.3629	4.3746	4.1881
3	3.39	3.2025	3.4066	3.5454	3.5553	3.4321
6	0.87	0.8695	0.8741	0.882	0.8821	0.8801
9	0.18	0.1791	0.1758	0.1758	0.1758	0.1771

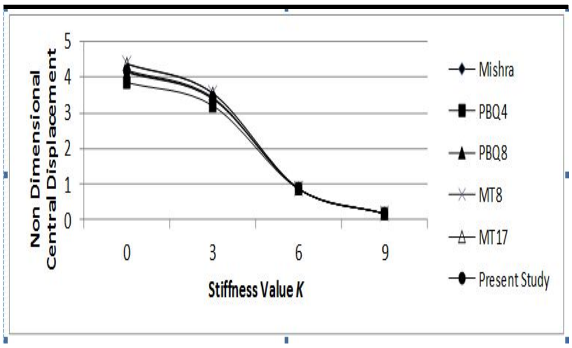

Figure 4. Non-dimensional central displacement of simply supported plates with different K values subjected to uniformly distributed load

Table 4.Non-dimensional central displacements for the simply supported plates with concentrated load

K	Mishra and Chakrabarti	O¨zgan and Dalo˘glu		Y.I. O¨zdemir		Present study
		PBQ4	PBQ8	MT8	MT17	
0	1.25	1.1578	1.246	1.2911	1.2844	1.2569
3	1.065	0.9968	1.0605	1.0896	1.082	1.06
6	0.427	0.4062	0.4197	0.421	0.4119	0.3845
9	0.214	0.2008	0.2063	0.2064	0.1973	0.1632

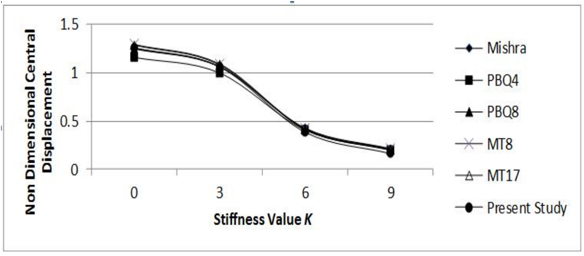

Figure 5. Non-dimensional central displacement of simply supported plates with different K values subjected to concentrated load

CONCLUSION

In this study, a four noded rectangular Kirchhoff's plate element with Winkler foundation integrated and having three degrees of freedom per node is developed for the analysis of plates resting on elastic foundation. The element is tested for different boundary conditions and different types of loads for different cases of elastic foundations and it gives satisfactory results comparing with exact classical solutions and results available from literature. It is seen that the above element can be used for the analysis of thin and moderately thick plates on Winkler foundation. The element is free from the problem of shear locking and having C^2 continuity. It gives more realistic deformed shape. Instead of using higher order finite elements which are more complex and requires more computational effort, this element is a better alternative as it is simple and requires less computational effort.

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