A STUDY IN MATHEMATICAL RELATION BETWEEN EIGEN VALUE AND SECOND ORDER CONTROL SYSTEM PARAMETER

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Abstract— In this paper generally generate relation between engineering mathematics(Eigen value) & second order control system. in engineering mathematics mainly focus on Eigen value theory, linear algebra, determinant & matrix. And second order control system mainly focus on time domain analysis second order system, state model single input single output(SISO) & multiple input multiple output(MIMO) control system.
Final result discussion the effect of Eigen value theorem in time domain analysis and find out The behavior of system. Finally mathematical relation between Eigen value and second order control system parameter from state model system to other systems.

Index Terms— Eigen value theory, state model, system matrix **etc.**

I. INTRODUCTION

The classical control theory and methods that we have been using in class to date are based on a simple input-output description of the plant, usually expressed as a transfer function. These methods do not use any knowledge of the interior structure of the plant, and limit us to single-input single-output (SISO) systems, and as we know it allows only limited control of the closed-loop behavior when feedback control is used[1], [2] . The concept of the state of a dynamic system refers to a minimum set of variables, known as state variables**.**

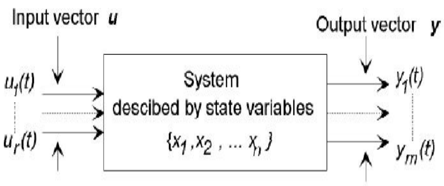

Fig1-System inputs and outputs.

State Equation Based Modeling Procedure

The complete system model for a linear time-invariant system consists of a set of n state equations, defined in terms of the matrices A and B, and a set of output equations that relate any output variables of interest to the state variables and inputs, and expressed in terms of the C and D matrices. The task of modeling the system is to derive the elements of the matrices, and to write the system model In case of theoretically a mathematical modal the dynamic of system three type of variable like input variable, output variable, and the state variable[1],[2].the state model of linear system equation

$\dot{x}(t)$ = Ax(t) + Bu(t). (1) This equation is called state equation.

Y(t) = Cx(t) + Du(t). (2) This equation is called output equation.

A = system matrix (n×n).
B = Input matrix (n×m).
C = Output matrix (p×1).
D= Transmission matrix (p×m).
X= input vector
Y= output vector

The matrices A and B are properties of the system and are determined by the system structure and elements. The output equation matrices C and D are determined by the particular choice of output variables.

Take Laplace transformation of equation (1) and (2) both side assume initial condition zero. And find out transfer function.

Transfer function (TF) of State Variable Systems

$=\frac{Y(S)}{U(S)}$

$= C(SI-A)^{-1}B + D.$

$= \frac{C\,adj(SI-A).B}{Det(SI-A)}+D.$ (3)

The above transfer function |SI-A| = 0 is called characteristic equation. In engineering math root of this equation is called Eigen value and in control system is called closed loop pole[3]. were A is n×n square matrix and [SI-A] is a singular matrix.

Standard form of transfer function in time domain second order system

$\frac{Y(S)}{U(S)}= \frac{w_n^2}{S^2+2\zeta w_n S+w_n^2}$ (4)

Denominator of Transfer function is a characterstic polynomial

II. TIME DOMAIN ANALYSIS

The time domain analysis transfer function (TF) of second order system and its characteristic equation compare with Eigen value theorem from mathematics and find the value of damping factor and undammed natural frequency[4]. After comparison produced relation between Eigen value and damping factor. and also relation between element of matrix [A] and damping ratio. By this relation find the nature of system like under damped, over damped, perfect damped, undammed etc.

Explanation between state model system and time domain second order system

If in the state model system matrix [A] change in the form of its adj[A],inverse of [A], $[A]^m$,k[A] etc. discuss the effect of in this form all parameter in time domain second order system like peak time, maximum overshoot, rise time, delay time etc[5].

Where ω_n = $\boldsymbol{natural\ frequency}$.

ζ = damping ratio.

EIGEN VALUE THEOREM

Denominator of transfer function equate to zero is called characteristic equation and Root of characteristic equation is called Eigen value or closed loop pole[6].

Mathematical characteristic equation

Let consider system matrix [A] of order 2×2 $|A-\lambda I|= 0$

Let $[A] = \begin{bmatrix}\lambda_1 & \lambda_2\\ \lambda_3 & \lambda_4\end{bmatrix}$ and $\lambda I = \begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}$.

$$\begin{vmatrix}\lambda_1 & \lambda_2\\ \lambda_3 & \lambda_4\end{vmatrix} - \begin{vmatrix}\lambda & 0\\ 0 & \lambda\end{vmatrix} = 0$$

$$\begin{vmatrix}\lambda_1-\lambda & \lambda_2\\ \lambda_3 & \lambda_4-\lambda\end{vmatrix} = 0$$

$$(\lambda_1-\lambda)(\lambda_4-\lambda) - \lambda_2\lambda_3 = 0$$

$$\lambda^2-(\lambda_1+\lambda_4)\lambda + (\lambda_1\lambda_4-\lambda_2\lambda_3) = 0 \quad (5)$$

It is a characteristic equation.

Let α and β are root of this equation.(5)

Then, sum of root $(\alpha+\beta) = \lambda_1+\lambda_4$. **And product of root** $(\alpha\beta) = (\lambda_1\lambda_4-\lambda_2\lambda_3)$.

Roots are $(\alpha , \beta) = \frac{(\lambda_1+\lambda_4)\pm\sqrt{(\lambda_1-\lambda_4)^2+4\lambda_2\lambda_3}}{2}$. By using above formula find eigen value of state variable system.

Now transfer function time domain analysis second order system characteristic equation.

$$S^2+2\zeta w_n S + w_n^2 = 0 \quad (6)$$

Let roots are α and β of equation (6) then sum of root $(\alpha+\beta) = -2\zeta w_n$ & product of root$(\alpha\beta) = w_n^2$

Now compare both above characteristic equation(5) and (6) find ζ & w_n.

$$\boldsymbol{\delta} = \frac{\alpha+\beta}{-2\sqrt{\alpha\beta}}.$$

$$\omega_n = \sqrt{\lambda_1\lambda_4-\lambda_2\lambda_3}.$$

$$\boldsymbol{\delta} = \frac{\lambda_1+\lambda_4}{-2\sqrt{\lambda_1\lambda_4-\lambda_2\lambda_3}}\ (7), \omega_n = \sqrt{\alpha\beta}$$

Where ζ &w_n both are positive.

$$\zeta \geq 0\ (8)\ \&\ \omega_n > 0$$

From equation (7) and (8)

$$\frac{\lambda_1+\lambda_4}{-2\sqrt{\lambda_1\lambda_4-\lambda_2\lambda_3}} \geq 0$$

$$\lambda_1+\lambda_4 \leq 0\ \&\ \lambda_1\lambda_4-\lambda_2\lambda_3 \geq 0$$

If above condition is not satisfied then state model system solution not possible[7].

Generate relation in Eigen value and following parameter second order system[6],[7].

$\omega_n = \sqrt{\alpha\beta}$ (7) from equation (6)

(a) $\omega_n \propto$ eigen value.

(b)Settling time$(T_s) = \frac{4}{\zeta\omega_n}$

Put the value of ω_n and ζ in equation **(**b)

$$T_S = \frac{4}{\frac{(\alpha+\beta)\sqrt{\alpha\beta}}{-2\alpha\beta}} = \frac{-8}{(\alpha+\beta)}.$$

Hence when in $\alpha\ and\ \beta$ any one increases then settling time

(c) Peak time$(T_p) = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$

Put the value of ω_n and ζ in equation (c)

$$\frac{\pi}{\sqrt{\alpha\beta}\sqrt{(1-(\frac{\alpha+\beta}{-2\sqrt{\alpha\beta}})^2}} = \frac{2\pi}{\sqrt{4\alpha\beta-(\alpha+\beta)^2}}$$

Only Valid for Under Damped system

(d) Rise time$(T_r) = \frac{\pi-\varphi}{\omega_d} = \frac{\pi-\cos^{-1}\zeta}{\omega_n\sqrt{1-\zeta^2}} = \frac{2\left\{\pi-\cos^{-1}-\frac{(\alpha+\beta)}{2\sqrt{\alpha\beta}}\right\}}{\sqrt{4\alpha\beta-(\alpha+\beta)^2}}$

(e) Delay time$(T_d) = \frac{1+.7\zeta}{\omega_n}$

$$= \frac{1+.7\frac{\alpha+\beta}{-2\sqrt{\alpha\beta}}}{\sqrt{\alpha\beta}} = \frac{2(\alpha\beta-0.7(\alpha+\beta))}{2\sqrt{\alpha\beta}}$$

Effect of Eigen value in damping natural (ζ**) undammed natural frequency** ω_n

$$\zeta = \frac{\alpha+\beta}{-2\sqrt{\alpha\beta}}.$$

If I have system matrix [A] then direct find out damping nature of system.

We Know that

$\zeta = 0$ system undamped.

$\zeta < 1$ *system under damped.*

$\zeta = 1$ system critical damped.

$\zeta > 1$ *system over damped.*

Discuss its all behavior in time domain second order system if change system matrix[A]. then change behavior of system.

Case[1]. compare if system matrix [A] and $[A]^T$.

Property of eigen value.

Eigen value of [A] = Eigen value of $[A]^T$

$$\zeta = \frac{\alpha+\beta}{-2\sqrt{\alpha\beta}}. \quad \text{and } \omega_n = \sqrt{\alpha\beta}$$

If Eigen value is not change then ζ and w_n are also not change. hance in time domain all parameter related to ζ and w_n are not effected.

Case[2]. compare if system matrix [A] and $[A]^{-1}$.

Property of Eigen value

If Eigen value of matrix [A] are $\alpha,\beta,\gamma\ldots\delta$ then eigen value of $[A]^{-1}$are also inverse

$\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma}\ldots\frac{1}{\delta}$

$$\zeta = \frac{\frac{1}{\alpha}+\frac{1}{\beta}}{-2\sqrt{\frac{1}{\alpha\beta}}} = \frac{\alpha+\beta}{-2\sqrt{\alpha\beta}}. \quad \text{and } \omega_n = \frac{1}{\sqrt{\alpha\beta}}$$

hence in this case ζ is not change but ω_n is inverse.

Case[3].Compare if system matrix [A] and k[A].

Property of Eigen value

If Eigen value matrix [A] are $\alpha, \beta, \gamma \ldots \delta$ then eigen value of k[A] are $k\alpha, k\beta, k\gamma \ldots k\delta$.

$$\zeta = \frac{k\alpha+k\beta}{-2\sqrt{k\alpha k\beta}} = \frac{\alpha+\beta}{-2\sqrt{\alpha\beta}}. \quad \text{and} \quad \omega_n = k\sqrt{\alpha\beta}$$

In this case ζ is not affected and ω_n **increases k times.**

Case[4]. **compare if system matrix [A] and $[A]^m$.**

Property of Eigen value

If Eigen value matrix [A] are $\alpha, \beta, \gamma \ldots \delta$ then Eigen value of$[A]^m$ are $\alpha^m, \beta^m, \gamma^m \ldots \delta^m$.

$$\zeta = \frac{\alpha^m+\beta^m}{-2\sqrt{\alpha^m\beta^m}}. \text{ And } \omega_n = \sqrt{\alpha^m\beta^m}$$

In this case ζ **and** ω_n **both are change**.

Case[5]. compare if system matrix [A] and $adj[A]$ only valid for nonsingular matrix [A].

Property of Eigen value

If Eigen value of matrix [A] are $\alpha, \beta, \gamma \ldots \delta$ then eigen value of adj[A] are $\frac{|A|}{\alpha}, \frac{|A|}{\beta}, \frac{|A|}{\gamma} \cdots\cdots\cdots, \frac{|A|}{\delta}$.

$$\zeta = \frac{\frac{|A|}{\alpha}+\frac{|A|}{\beta}}{-2\sqrt{\frac{|A|}{\alpha}\frac{|A|}{\beta}}} = \frac{\alpha+\beta}{-2\sqrt{\alpha\beta}}. \text{ and } \omega_n = \sqrt{\frac{|A|}{\alpha}\frac{|A|}{\beta}} = |A|\sqrt{\frac{1}{\alpha\beta}}.$$

In this case damping factor ζ is not affected but ω_n is affected .

CONCLUSION

In conclusion mainly second order system generate mathematical equation between 'natural frequency and Eigen value' or 'damping ratio and Eigen value'. Again square root of determinant of system matrix called natural frequency. If any Eigen value increases then settling time also increases. and generate mathematical equation between Eigen value and all parameter of second order system like settling time, delay time, peak time and rise time etc. and by this equation define nature of system like over damped, undammed, under damped, critical damped and stability etc.

If system matrix [A] is change in following form of like $[A]^{-1}$,k[A], adj[A],then damping factor are not change but natural frequency are change but in case of $[A]^m$ damping factor and natural frequency both are change but if in case of $[A]^T$ then damping factor and natural frequency both are not change.

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