PERFORMANCE ANALYSIS OF AN $M/G/1$ RETRIAL QUEUE WITH BALKING AND WORKING VACATION MODEL

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Abstract— This paper deals with a single server retrial queueing system with balking and working vacation in which the server works with different service rates rather than completely terminating the service during its vacation period. We assume that both service times in a regular service period and in a working vacation period are generally distributed. The steady state distributions of the server state and the number of jobs in the orbit were obtained along with other performance measures.

Keywords— Steady state, Retrial queue, Balking, Single working vacation.

I. INTRODUCTION

The queueing system is a powerful tool for modeling communication networks, transportation networks, production lines and operating systems. In recent years, computer networks and data communication systems are the fastest growing technologies, which have led to significant development in applications such as swift advance in internet, audio data traffic, video data traffic, etc.
The retrial queueing system is characterized by an arriving job which finds the server busy, leaves the service area and repeats its demand after some time. Between trials, the blocked job joins a pool of unsatisfied jobs called orbit, for example, web access, telecommunication networks, packet switching networks, collision avoidance, star local area networks, etc.
The server works continuously as long as there is at least one job in the system. When the server finishes serving a job and finds the system empty, it leaves the system for a period of time called a vacation. This is seen in maintenance activities such as telecommunication networks, customized manufacturing systems, production systems, etc.
Arivudainambi and Godhandaraman (2015) analyzed a single server retrial queueing system with general repeated attempts, balking, second optional service and single vacation. Arivudainambi and Godhandaraman (2014) widely discussed a single server retrial queueing system with general repeated attempts and single working vacation. Artalejo (2010) have concluded explicit surveys on retrial queueing systems.
Servi and Finn (2002) analyzed an M/M/1 queue with working vacation and it was extended by Wu and Takagi (2006) for an M/G/1 queue with working vacation, where both service time in the service period and in the working vacation are generally distributed. Do (2010) introduced a M/M/1 retrial queue with working vacations which was motivated by the performance analysis of a Media Access Control (MAC) function in wireless networks. Lot of work has been done in the literature for various queueing systems but not for the retrial queueing systems with balking and working vacation using the supplementary variable technique.
The rest of the paper is organized as follows. We have given a practical justification for the models in section 2 and a brief mathematical description of model is given in section 3. Section 4 deals with the derivations of the steady state distribution of the server. The mean number of jobs in the system and several performance measures are discussed in section 5. In section 6, some important special cases of this model are briefly discussed.

II. PRACTICAL JUSTIFICATION OF THE SUGGESTED MODEL

The suggested model has potential application in the transfer model of an email system. Simple mail transfer protocol (SMTP) is used to deliver the messages between the mail servers. On a remote machine, a mail transfer program contacts a server for TCP connection. When the TCP is connected, SMTP allows the sender to identify itself, specifies the recipient and then transfers a email message.
When the sender deposits the email in his/her own mail server, the mail server repeats continuously (retrial) until the message is delivered. In the mail server, the contact message follows the Poisson process. At the arrival epoch, the arriving message starts its service immediately if the server is free or else joins the buffer. Whereas, if the mail server is free, then the arriving message starts its service immediately.
In the buffer, each message waits for some amount of time and retries the service again. The mail server employs a spam filter service in a low service rate to prevent spam mails, this is done to filter the incoming message via the normal mail receiving service. To keep the mail server functioning well, some maintenance activities are needed.

For example, the virus scan is an important maintenance activity in the mail server. When the server is idle, it is performed. During the period of virus scan, the server can still provide its service, but with lower processing speed. When the virus scan is done, the server enters the idle state and waits the for requests to arrive. In this scenario, the buffer in the sender mail server, the receiver mail server, the retransmission policy, and the maintenance activities correspond to the orbit, the server, the retrial and the single working vacation policy respectively in the queueing terminology.

III. MODEL DESCRIPTION & ERGODICITY CONDITION

A job arrives for service according to the Poisson process at rate λ and start its service immediately if the server is available. If an arriving job finds the server busy, then the job joins the orbit with probability p or balk the system with probability q. The job from the orbit to the server is governed by an arbitrary law with distribution function $R(t)$ and Laplace-Stieltjes transform (LST) $R^*(\theta)$.

In a regular service period, the jobs are served at a service rate μ_b follows a random variable with distribution function $S_b(t)$ and Laplace-Stieltjes transform $S_b^*(\theta)$. When the orbit becomes empty, the server begins single working vacation.

During a working vacation period, the jobs are served at a lower service rate μ_v follows a random variable with distribution function $S_v(t)$ and Laplace-Stieltjes transform $S_v^*(\theta)$. After completion of the single working vacation, the server changes service rate from μ_v to μ_b, if there are jobs in the orbit.

The state of the system at time t can be defined by the Markov process $\{N(t);\ t \geq 0\} = \{(C(t), X(t), \xi_0(t), \xi_1(t), \xi_2(t), t \geq 0\}$, where $C(t)$ denotes the server state (0,1 and 2, according to the server being free, busy and working vacation respectively) and $X(t)$ is the number of jobs in the orbit at time t. If $C(t) = 0$ and $X(t) > 0$, then $\xi_0(t)$ represents the elapsed retrial time, if $C(t) = 1$, then $\xi_1(t)$ represents the elapsed service time during a regular busy period at time t, if $C(t) = 2$ and $X(t) \geq 0$, then $\xi_2(t)$ represents the elapsed working vacation at time t.

The functions $\theta(x)dx, \mu_b(x)dx$ and $\mu_v(x)dx$ are the conditional completion rates for repeated attempts, service and working vacation respectively at time x. i.e.,

$\theta(x)dx = \frac{dR(x)}{1-R(x)},\ \mu_b(x)dx = \frac{dS_b(x)}{1-S_b(x)}, \mu_v(x)dx = \frac{dS_v(x)}{1-S_v(x)}.$

3.1 . Ergodicity Condition

We analyze the ergodicity of the embedded Markov chain at departure/vacation epochs. Let $\{t_n;\ n \in N\}$ be the sequence of epochs of either service completion times or vacation termination time. The sequence of random vectors $Z_n = \{C(t_n +),\ X(t_n +)\}$ form a Markov chain which is the embedded Markov chain for the queueing system. Its state space is $S = \{0, 1 \text{ and } 2\} \times N$.

Theorem 1: The embedded Markov chain $\{Z_n;\ n \in N\}$ is ergodic if and only if $\lambda E(S_b) < R^*(\lambda)$.

Proof: We know that $\{Z_n;\ n \in N\}$ is an irreducible and aperiodic Markov chain. Foster's criterion was used to prove the sufficient condition of ergodicity. An irreducible and aperiodic Markov chain is ergodic if there exists a non-negative function $f(j), j \in N$ and $\epsilon > 0$ such that the mean drift $\chi_j = E[f(z_{n+1}) - f(z_n)|z_n = j]$ is finite for all $j \in N$ and $\chi_j \leq -\epsilon$ for all $j \in N$, except perhaps for some finite number j. We consider the function $f(j) = j$, then we have $\chi_j = \begin{cases} \lambda E(S_b) - R^*(\lambda), & j = 1, 2, \ldots \\ \lambda E(S_b) - 1, & j = 0 \end{cases}$.

The inequality $\lambda E(S_b) < R^*(\lambda)$ is a sufficient condition for ergodicity. The same inequality is also necessary for ergodicity. Kaplan's condition is used to prove the necessary condition of ergodicity. Sennot et al. (1983) states that non-ergodicity of the Markov chain $\{Z_n;\ n \geq 1\}$ satisfies Kaplan's condition, namely $\chi_j < \infty$ for all $j \geq 0$ and there exists $j_0 \in N$ such that $\chi_j \geq 0$ for $j \geq j_0$. In our case, Kaplan's condition is satisfied because there exists h such that $r_{ij} = 0$ for $j < i - h$ and $i > 0$, where $R = (r_{ij})$ is the one step transition matrix of $\{Z_n;\ n \geq 1\}$. Then, the inequality $\lambda E(S_b) \geq R^*(\lambda)$ implies the non-ergodicity of the Markov chain.

IV. STEADY STATE DISTRIBUTION OF THE SERVER STATE

For the process $\{N(t), t \geq 0\}$, the probabilities are define as

$$P_0(t) = P\{C(t) = 0, X(t) = 0\}$$
$$P_n(x,t)dx = P\{C(t) = 0, X(t) = n, x \leq \xi_0(t) < x + dx\}, \text{ for } t \geq 0, x \geq 0 \text{ and } n \geq 1$$
$$Q_{n,b}(x,t)dx = P\{C(t) = 1, X(t) = n,\ x \leq \xi_1(t) < x + dx\}, \text{ for } t \geq 0, x \geq 0 \text{ and } n \geq 0$$
$$Q_{n,v}(x,t)dx = P\{C(t) = 2, X(t) = n,\ x \leq \xi_2(t) < x + dx\}, \text{ for } t \geq 0, x \geq 0 \text{ and } n \geq 0$$

We assume that the steady state condition $\lambda E(S_b) < R^*(\lambda)$ is fulfilled, hence we can set $P_0 = \lim_{t\to\infty} P_0(t)$, $P_n(x) = \lim_{t\to\infty} P_n(t,x)$ for $x \geq 0, n \geq 1$, $Q_{n,b}(x) = \lim_{t\to\infty} Q_{n,b}(t,x)$ for $x \geq 0, n \geq 0$ and $Q_{n,v}(x) = \lim_{t\to\infty} Q_{n,v}(t,x)$ for $x \geq 0, n \geq 0$

By the method of supplementary variables, we obtain the following steady state balance equations

$$\lambda P_0 = \int_0^\infty Q_{0,v}(x)\,\mu_v(x)dx \quad (1)$$
$$\frac{d}{dx}P_n(x) + [\lambda + \theta(x)]P_n(x) = 0, x > 0, n \geq 1 \quad (2)$$
$$\frac{d}{dx}Q_{0,b}(x) + [\lambda + \mu_b(x)]Q_{0,b}(x) = \lambda\, q Q_{0,b}(x), x > 0 \quad (3)$$
$$\frac{d}{dx}Q_{n,b}(x) + [\lambda + \mu_b(x)]Q_{n,b}(x) = \lambda\, p\, Q_{n-1,b}(x) + \lambda\, q\, Q_{n,b}(x), x > 0, n \geq 1 \quad (4)$$
$$\frac{d}{dx}Q_{0,v}(x) + [\lambda + \mu_v(x)]Q_{0,v}(x) = \lambda\, q Q_{0,v}(x), x > 0 \quad (5)$$
$$\frac{d}{dx}Q_{n,v}(x) + [\lambda + \mu_v(x)]Q_{n,v}(x) = \lambda\, p\, Q_{n-1,v}(x) + \lambda\, q\, Q_{n,v}(x), x > 0, n \geq 1 \quad (6)$$

The above set of equations can be solved using the steady state boundary conditions at $x = 0$,

$$P_n(0) = \int_0^\infty Q_{n,v}(x)\,\mu_v(x)dx + \int_0^\infty Q_{n,b}(x)\,\mu_b(x)dx, n \geq 1 \quad (7)$$
$$Q_{0,b}(0) = \lambda P_0 + \int_0^\infty P_1(x)\theta(x)dx \quad (8)$$
$$Q_{n,b}(0) = \int_0^\infty P_{n+1}(x)\theta(x)dx + \lambda \int_0^\infty P_n(x)\theta(x)dx, n \geq 1 \quad (9)$$
$$Q_{0,v}(0) = \int_0^\infty Q_{0,b}(0)\mu_b\,(x)dx \quad (10)$$

The normalization condition is given by

$$P_0 + \sum_{n=1}^\infty \int_0^\infty P_n(x)dx + \sum_{n=0}^\infty \int_0^\infty Q_{n,b}(x)dx \quad + \sum_{n=0}^\infty \int_0^\infty Q_{n,v}(x)dx = 1 \quad (11)$$

Let us define the probability generating functions as

$P(x,z) = \sum_{n=1}^\infty z^n P_n(x)$, $P(0,z) = \sum_{n=1}^\infty z^n P_n(0)$, $Q_b(x,z) = \sum_{n=0}^\infty z^n Q_{n,b}(x)$, $Q_b(0,z) = \sum_{n=0}^\infty z^n Q_{n,b}(0)$, $Q_v(x,z) = \sum_{n=0}^\infty z^n Q_{n,v}(x)$, $Q_v(0,z) = \sum_{n=0}^\infty z^n Q_{n,v}(0)$ for $|z| \leq 1$ and $x > 0$.

Theorem 2.

Under the stability condition $\lambda\, E(S_b) < R^*(\lambda)$, the stationary distributions of the number of jobs in the system when the server being idle, busy and on working vacation are

$$P(z) = \frac{P_0 z[1-R^*(\lambda)]\{[1-S_v^*(\lambda\, p(1-z))]+S_v^*(\lambda\, p)[1-S_b^*(\lambda\, p(1-z))]\}}{\{S_v^*(\lambda\, p)\{[z+(1-z)R^*(\lambda)]S_b^*(\lambda\, p(1-z))-z\}\}} \quad (12)$$
$$Q_b(z) = \frac{\{P_0[1-S_b^*(\lambda\, p(1-z))]\{[1-S_v^*(\lambda\, p(1-z))][z+(1-z)R^*(\lambda)]+(1-z)R^*(\lambda)S_v^*(\lambda\, p)\}\}}{\{S_v^*(\lambda\, p)(1-z)p\{[z+(1-z)R^*(\lambda)]S_b^*(\lambda\, p(1-z))-z\}\}} \quad (13)$$
$$Q_v(z) = \frac{\{P_0[1-S_v^*(\lambda\, p(1-z))]\}}{\{S_v^*(\lambda\, p)(1-z)p\}} \quad (14)$$
$$P_0 = \frac{\{S_v^*(\lambda p)[R^*(\lambda)-\lambda p E(S_b)]\}}{\{\lambda\, pE(S_v)+R^*(\lambda)S_v^*(\lambda\, p)+q[\lambda\, E(S_v)R^*(\lambda)+\lambda\, E(S_b)R^*(\lambda)S_v^*(\lambda\, p)]\}} \quad (15)$$

Proof.

Multiplying equations $(2) - (6)$ by suitable powers of z and summing over n, we obtain the following partial differential equations

$$\frac{\partial}{\partial x}P(x,z) + [\lambda + \theta\,(x)]P(x,z) = 0 \quad (16)$$
$$\frac{\partial}{\partial x}Q_b(x,z) + [\lambda\, p(1-z) + \mu_b(x)]Q_b(x,z) = 0 \quad (17)$$
$$\frac{\partial}{\partial x}Q_v(x,z) + [\lambda\, p(1-z) + \mu_v(x)]Q_v(x,z) = 0 \quad (18)$$

Solving the above partial differential equations $(16) - (18)$, we get

$$P(x,z) = P(0,z)[1-R(x)]e^{-\lambda\, x} \quad (19)$$
$$Q_b(x,z) = Q_b(0,z)[1-S_b(x)]e^{-\lambda\, p(1-z)\, x} \quad (20)$$
$$Q_v(x,z) = Q_v(0,z)[1-S_v(x)]e^{-\lambda\, p(1-z)\, x} \quad (21)$$

Multiplying equation (7) by suitable powers of z, summing over n from 1 to ∞ and after some algebraic simplification we arrive,

$$P(0,z) = \int_0^\infty Q_v(x,z)\,\mu_v(x)dx + \int_0^\infty Q_b(x,z)\,\mu_b(x)dx - Q_{0,v}\,(0) - \lambda\, P_0 \quad (22)$$

Multiplying equations $(8) - (10)$ by appropriate powers of z, summing over n from 0 to ∞ and after some algebraic manipulation, we obtain

$$Q_b(0,z) = \frac{1}{z}\int_0^\infty P(x,z)\,\theta(x)dx + \lambda \int_0^\infty P(x,z)\,dx + \lambda\, P_0 \quad (23)$$
$$Q_v(0,z) = Q_{0,v}\,(0) \quad (24)$$

From equation (5), we get

$$Q_{0,v}\,(x) = Q_{0,v}\,(0)[1-S_v(x)]e^{-\lambda\, px} \quad (25)$$

Multiplying equation (25) by $\mu_v(x)$ on both sides and integrating with respect to x from 0 to ∞ and using equation (1), we have

$$Q_{0,v}\,(0) = \frac{\lambda\, P_0}{S_v^*(\lambda\, p)} \quad (26)$$

Substituting equation (26) in (24), we obtain

$$Q_v(0,z) = \frac{\lambda\, P_0}{S_v^*(\lambda\, p)} \quad (27)$$

Using equation (19) in equation (23), we get

$$Q_b(0,z) = \lambda P_0 + \left(\frac{z+(1-z)R^*(\lambda)}{z}\right) P(0,z) \quad (28)$$

Further using equations $(20) - (21)$, and (26) in equation (22), we get

$$P(0,z) = Q_v(0,z)S_v^*(\lambda p(1-z)) + Q_b(0,z)S_b^*(\lambda p(1-z)) - \frac{\lambda P_0}{S_v^*(\lambda p)} - \lambda P_0 \quad (29)$$

Using the equations (27) and (28), we get

$$P(0,z) = \frac{\lambda z P_0}{S_v^*(\lambda p)}\left\{\frac{[1-S_v^*(\lambda p(1-z))]+S_v^*(\lambda p)[1-S_b^*(\lambda p(1-z))]}{S_v^*(\lambda p(1-z))[z+(1-z)R^*(\lambda)]-z}\right\} \quad (30)$$

Substituting equation (30) in (28), we get

$$Q_b(0,z) = \left[\frac{\lambda P_0\{[1-S_v^*(\lambda p(1-z))]+S_v^*(\lambda p)[1-S_b^*(\lambda p(1-z))]\}}{S_v^*(\lambda p)\{S_v^*(\lambda p(1-z))[z+(1-z)R^*(\lambda)]-z\}}\right][z + (1-z)R^*(\lambda)] + \lambda P_0 \quad (31)$$

Substituting equations $(30) - (31)$ and (27) in equations $(19) - (21)$ and after some algebraic manipulation, we obtain,

$$P(x,z) = \left[\frac{\lambda P_0 z\{[1-S_v^*(\lambda p(1-z))]+S_v^*(\lambda p)[1-S_b^*(\lambda p(1-z))]\}}{S_v^*(\lambda p)\{S_v^*(\lambda p(1-z))[z+(1-z)R^*(\lambda)]-z\}}\right][1-R(x)]e^{-\lambda x}$$

$$Q_b(x,z) = \left\{\left[\frac{\lambda P_0\{[1-S_v^*(\lambda p(1-z))]+S_v^*(\lambda p)[1-S_b^*(\lambda p(1-z))]\}}{S_v^*(\lambda p)\{S_v^*(\lambda p(1-z))[z+(1-z)R^*(\lambda)]-z\}}\right][z + (1-z)R^*(\lambda)] + \lambda P_0\right\} \times [1-S_b(x)]e^{-\lambda p(1-z)x}$$

$$Q_v(x,z) = \frac{\lambda P_0}{S_v^*(\lambda p)}[1-S_v(x)]e^{-\lambda p(1-z)x}$$

Integrating the above equations with respect to x from 0 to ∞, we finally get the required results $(12) - (14)$. At this point, the only unknown is P_0, which can be determined using the normalization condition $P_0 + P(1) + Q_b(1) + Q_v(1) = 1$. Let $K(z) = P_0 + P(z) + z[Q_b(z) + Q_v(z)]$ be the probability generating function for the number of jobs in the system and $H(z) = P_0 + P(z) + Q_b(z) + Q_v(z)$ be the probability generating function for the number of jobs in the orbit at stationary point of time. Thus, we have the following theorem.

Theorem 3.
Under the stability condition $\lambda E(S_b) < R^*(\lambda)$, the probability generating function of the system size and orbit size distribution at stationary point of time is

$$K(z) = \frac{P_0\left\{\{[1-S_v^*(\lambda p(1-z))][z+(1-z)R^*(\lambda)]+(p-z)R^*(\lambda)S_v^*(\lambda p)\}S_b^*(\lambda p(1-z))+qz\{R^*(\lambda)S_v^*(\lambda p)-[1-S_v^*(\lambda p(1-z))][1-R^*(\lambda)]\}\right\}}{p\, S_v^*(\lambda p)\{S_b^*(\lambda p(1-z))[z+(1-z)R^*(\lambda)]-z\}} \quad (32)$$

$$H(z) = \frac{P_0\{[1-S_v^*(\lambda p(1-z))][pz+(1-pz)R^*(\lambda)]+(1-pz)R^*(\lambda)S_v^*(\lambda p)-qS_b^*(\lambda p(1-z))R^*(\lambda)S_v^*(\lambda p)\}}{p\, S_v^*(\lambda p)\{S_b^*(\lambda p(1-z))[z+(1-z)R^*(\lambda)]-z\}} \quad (33)$$

where P_0 is given in (15).

V. PERFORMANCE MEASURES

In this section, we obtain some performance measures for the system under steady state. Let I be the steady state probability that the server is idle during the retrial time, U be the steady state probability that the server is busy, R be the steady state probability that the server is on working vacation, D be the steady state probability that the server is idle or on working vacation, R_0 be the steady state probability that the system is empty while the server is on working vacation, E be the steady state probability that the system is empty and J be the steady state probability that the orbit is empty.

$$I = P(1) = \frac{[1-R^*(\lambda)][\lambda pE(S_v)+S_v^*(\lambda p)\lambda pE(S_b)]}{[\lambda pE(S_v)+R^*(\lambda)S_v^*(\lambda p)+q\{\lambda E(S_v)R^*(\lambda)+\lambda E(S_b)R^*(\lambda)S_v^*(\lambda p)\}]}$$

$$U = Q_b(1) = \frac{\lambda E(S_b)[\lambda pE(S_b)+R^*(\lambda)S_v^*(\lambda p)]}{[\lambda pE(S_v)+R^*(\lambda)S_v^*(\lambda p)+q\{\lambda E(S_v)R^*(\lambda)+\lambda E(S_b)R^*(\lambda)S_v^*(\lambda p)\}]}$$

$$R = Q_v(1) = \frac{\lambda E(S_v)[R^*(\lambda)-\lambda pE(S_b)]}{[\lambda pE(S_v)+R^*(\lambda)S_v^*(\lambda p)+q\{\lambda E(S_v)R^*(\lambda)+\lambda E(S_b)R^*(\lambda)S_v^*(\lambda p)\}]}$$

$$D = P_0 + P(1) + Q_v(1) = \frac{[1-R^*(\lambda)][\lambda pE(S_v)+\lambda pE(S_b)S_v^*(\lambda p)]}{[\lambda pE(S_v)+R^*(\lambda)S_v^*(\lambda p)+q\{\lambda E(S_v)R^*(\lambda)+\lambda E(S_b)R^*(\lambda)S_v^*(\lambda p)\}]}$$

$$R_0 = Q_v(0) = \frac{[1-S_v^*(\lambda p)][R^*(\lambda)-\lambda pE(S_b)]}{p[\lambda pE(S_v)+R^*(\lambda)S_v^*(\lambda p)+q\{\lambda E(S_v)R^*(\lambda)+\lambda E(S_b)R^*(\lambda)S_v^*(\lambda p)\}]}$$

$$E = P_0 + R_0 = \frac{[R^*(\lambda)-\lambda pE(S_b)][1-qS_v^*(\lambda p)]}{p[\lambda pE(S_v)+R^*(\lambda)S_v^*(\lambda p)+q\{\lambda E(S_v)R^*(\lambda)+\lambda E(S_b)R^*(\lambda)S_v^*(\lambda p)\}]}$$

$$J = P_0 + R_0 + Q_0 = \frac{[R^*(\lambda)-\lambda pE(S_b)][1-qS_v^*(\lambda p)S_b^*(\lambda p)]}{pS_b^*(\lambda p)[\lambda pE(S_v)+R^*(\lambda)S_v^*(\lambda p)+q\{\lambda E(S_v)R^*(\lambda)+\lambda E(S_b)R^*(\lambda)S_v^*(\lambda p)\}]}$$

Differentiating equation (32) with respect to z and evaluating at $z = 1$, the mean number of jobs in the system is obtained as,

$$L_s = \frac{[\lambda^2p^2E(S_b^2)+2\lambda pE(S_b)(1-R^*(\lambda))]}{2[R^*(\lambda)-\lambda pE(S_b)][\lambda pE(S_v)+R^*(\lambda)S_v^*(\lambda p)+q\{\lambda E(S_v)R^*(\lambda)+\lambda E(S_b)R^*(\lambda)S_v^*(\lambda p)\}]}$$
$$\times [\lambda E(S_v) + R^*(\lambda)S_v^*(\lambda p) + q\{\lambda E(S_b)R^*(\lambda)S_v^*(\lambda p) + \lambda E(S_v)(R^*(\lambda)-1)\}]$$
$$+ \frac{\lambda^2p^2E(S_v^2)+2\lambda pE(S_v)[1-R^*(\lambda)]+2\lambda^2p^2E(S_v)E(S_b)+2\lambda pE(S_b)R^*(\lambda)S_v^*(\lambda p)}{2p[\lambda pE(S_v)+R^*(\lambda)S_v^*(\lambda p)+q\{\lambda E(S_v)R^*(\lambda)+\lambda E(S_b)R^*(\lambda)S_v^*(\lambda p)\}]}$$

$$+\frac{q[\lambda^2 p^2\, E(S_b^2)R^*(\lambda)S_v^*(\lambda\, p)+2\lambda\, pE(S_v)(R^*(\lambda)-1)+\lambda^2 p^2 E(S_v^2)(R^*(\lambda)-1)]}{2p[\lambda\, pE(S_v)+R^*(\lambda)S_v^*(\lambda\, p)+q\{\lambda\, E(S_v)R^*(\lambda)+\lambda\, E(S_b)R^*(\lambda)S_v^*(\lambda\, p)\}]}$$

The differentiating equation (33) with respect to z and evaluating at $z = 1$ and the mean number of jobs in the orbit is obtained as,

$$L_q = \frac{\lambda^2 p^2 E(S_b^2)+2\lambda\, pE(S_b)[1-R^*(\lambda)]}{2[R^*(\lambda)-\lambda\, pE(S_b)]} + \frac{\lambda^2 p^2 E(S_v^2)+2\lambda\, pE(S_v)[1-R^*(\lambda)]}{2[\lambda\, pE(S_v)+R^*(\lambda)S_v^*(\lambda\, p)+q\{\lambda\, E(S_v)R^*(\lambda)+\lambda\, E(S_b)R^*(\lambda)S_v^*(\lambda p)\}]}$$
$$+\frac{q[\lambda^2\, pE(S_b^2)R^*(\lambda)S_v^*(\lambda\, p)+\lambda^2 pE(S_v^2)R^*(\lambda)]}{2[\lambda\, pE(S_v)+R^*(\lambda)S_v^*(\lambda\, p)+q\{\lambda\, E(S_v)R^*(\lambda)+\lambda\, E(S_b)R^*(\lambda)S_v^*(\lambda p)\}]}$$

VI. SPECIAL CASES

In this section, we analyze briefly some special cases of our model, which are consistent with the existing literature.

Case 1: If $R^*(\lambda) \rightarrow 1$, our model is reduced to the $M/G/1$ queue with single working vacation and the results agree with Zhang and Hou (2012).

Case 2: If $R^*(\lambda) \rightarrow 1$ and $S_v^*(\lambda) = 1$, our model is reduced to the $M/G/1$ queueing system and the results agree with Gross and Harris (1998).

CONCLUSION

In this paper, we introduced a single server retrial queueing system with general repeated attempts and balking and single working vacation. For this model, explicit expressions were obtained for the probability generating function of the server states, the number of jobs in a system, and the orbit were found using the supplementary variable technique. Various performance measures and special cases have been analyzed.

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